Magnetic Helicity Generation from the Cosmic Axion Field

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The coupling between a primordial magnetic field and the cosmic axion field generates a helical component of the magnetic field around the time in which the axion starts to oscillate. If the energy density of the seed magnetic field is comparable to the energy density of the universe at that time, then the resulting magnetic helicity is about $|H_B| \simeq (10^{-20}\text{G})^2\,\text{kpc}$ and remains constant after its generation. As a corollary, we find that the standard properties of the oscillating axion remain unchanged even in the presence of very strong magnetic fields.

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I. INTRODUCTION

The observations of galaxies and galaxy clusters unambiguously show that we live in a magnetized universe. Large-scale magnetic fields with coherence length $\xi \sim$ Mpc and intensities $B \sim \mu$G have been observed in all gravitationally bound large-scale structures. However, whether these magnetic fields have a primordial origin or were generated during the period of galaxy formation is still an open question (for a full discussion see Refs. [1, 2, 3, 4]). Understanding this point is a major goal of modern astrophysics and cosmology, and for this purpose it is extremely important to consider all the phenomena which leave a different *signature* on magnetic fields of different origin.

As we are going to show below, one of these phenomena is the interaction of the magnetic field with a cosmic axion field. We will show in particular that the primordial axion oscillations generate a helical component of the magnetic field. Of course, this effect regards only magnetic fields of primordial origin, since it takes place in the very short interval of time when axions start to oscillate. This happens when the temperature of the universe is about 1 GeV, much before the galaxy formation. In other words, if we believe in the existence of axions then any *primordial* magnetic field observed today must be (at least partially) *helical*.

The magnetic helicity is a very peculiar quantity associated with a magnetic field [5], and speculations about the generation of primordial helical magnetic fields exist in the literature [6, 7, 8, 9, 10, 11, 12, 13, 14]. In general, given an electromagnetic field $A^\mu = (A^0, \mathbf{A})$ we define the magnetic helicity as

$$H_B(t) = \int_V d^3x\, \mathbf{A} \cdot \nabla \times \mathbf{A}, \tag{1}$$

which can be considered as a Chern-Simons term because of the relation $\frac{1}{4}\int_{t_1}^{t_2} d^4x F_{\mu\nu}\widetilde{F}^{\mu\nu} = H_B(t_2) - H_B(t_1)$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field strength tensor, $\widetilde{F}^{\mu\nu} = (1/2\sqrt{-g})\,\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ its dual, $g = \det||g_{\mu\nu}||$ being the determinant of the metric tensor and $\epsilon^{\mu\nu\rho\sigma}$ the Levi-Civita (pseudo-)tensor. The definition of the helicity given above is valid in the Minkowski as well as in the expanding Freedman-Robertson-Walker universe, since it is easy to see that Eq. (1) is invariant under general coordinate transformations. On the other hand, it is odd under discrete P and CP transformations (it is a pseudo-scalar quantity). This leads to the well known result that the presence of magnetic helicity in our universe is a manifestation of a macroscopic P and CP violation.

Of course, the phenomenon we are investigating crucially depends on the existence of the axion field. This is related to the so-called Peccei-Quinn (PQ) mechanism [15] for the explanation of the smallness (or absence) of the $CP-$violating part of the QCD Lagrangian. After almost 30 years, this is still the most trusted argument for the solution of the above problem, known as the strong$-CP$ problem (for a general review see, e.g., Ref. [16]). This explains why the existence of the axion is largely believed, even without any experimental evidence for it, and justifies the great interest in its phenomenology.

Axions are chargeless, weakly interacting, pseudo-scalar particles which emerge as (pseudo-)Goldstone modes of the (almost) conserved PQ symmetry $U(1)_{\text{PQ}}$. Although chargeless, axions interact with photons by means of the anomalous term $g_{a\gamma}aF\widetilde{F}$, where $g_{a\gamma} = \alpha_{\text{em}}/(2\pi f_a)$, α_{em} is the electromagnetic fine structure constant, and f_a is the scale at which $U(1)_{\text{PQ}}$ is spontaneously broken, known as the axion or PQ constant. This scale characterizes all the axion properties on the phenomenological ground. Even though its value is not predicted by the PQ mechanism, and thus is model dependent, it is constrained to the very narrow allowed window $10^{10} \lesssim f_a \lesssim 10^{12}$ GeV by astrophysical and cosmological observations and considerations [17, 18]. This is a remarkable point since it will make all our results essentially independent on the specific axion model.

It is quite interesting (and even rather surprising) that the axion field, introduced to solve the strong$-CP$ problem in the QCD Lagrangian, generates a helical compo-

nent of the primordial magnetic field, contributing to the macroscopic $CP-$violation in the universe. This point can be more easily understood considering the cosmological evolution of the axion field. Axions are born when the universe is at the temperature $T \sim f_a$, and $U(1)_{\rm PQ}$ spontaneously breaks. After the universe has cooled down enough for the axion mass to be comparable with the Hubble expansion rate H, axions start to oscillate driving the QCD Lagrangian to its $CP-$conserving minimum. During that time axions constitute a (quasi-)zero momentum condensate, weakly interacting with the external magnetic field. The main effect of this external field is to "extract" axions from the condensate. This process is governed by the simple Eq. (28), where n_a is the number of axions per co-moving volume and ε is a positive, time dependent function, proportional to the magnetic energy, and inversely proportional to the conductivity of the primordial plasma. For vanishing magnetic fields, the number of axion per co-moving volume remains constant during the time of coherent oscillations. This is a well known result (see, e.g., Ref. [19] and references therein) which leads to the upper limit on the PQ constant (the so-called cosmological bound [20]). We will see later that this bond remains essentially unchanged even in the very strong field limit, owing to the smallness of ε.

However, the effect of the axion primordial oscillations on the magnetic field is more interesting. In fact, the evaporation of axions from the condensate is followed by the productions of photons and a resulting increment of the magnetic field strength. Since the probabilities of creating left-handed and right-handed photons are different (CP is not exact), the final result is an overproduction of photons of one kind with respect to the other. In other words, the helicity of the system changes.

More specifically, the axion coherent oscillations play the role of an α^2-dynamo on the evolution of the magnetic field because of the term $\alpha_{\rm dyn}\nabla\times\mathbf{B}$ [see Eq. (6)], where $\alpha_{\rm dyn}$ is proportional to ε. This term is P and CP odd, and is responsible for the generation of the magnetic helicity. As we expect from the above discussion, for this case the dynamo is not efficient ($\alpha_{\rm dyn}$ is small), owing mainly to the smallness of the axion-photon coupling $g_{a\gamma}$ and to the large value of the primordial conductivity σ. This last point reflects the known result that helicity is conserved when the conductivity is infinitively large.

The interaction between pseudo-scalar and electromagnetic fields has been the object of various papers in the literature.
In the seminal paper by Turner and Widrow [6], it was suggested that, during the inflationary epoch, small fluctuating magnetic fields could have been amplified due to the coupling with the axion field. This possibility was exhaustively studied in the subsequent papers by Garretson et al. [7], and Field and Carroll [8].
Ahonen et al. [21] studied the evolution of the cosmic axion field in the background of an external homogeneous magnetic field just after the QCD phase transition, while in a recent paper, Lee et al. [22] investigated the possibility of generating magnetic fields through the coupling of an evolving pseudo-scalar field during the period of the Large Scale Structures formation.

In the next Section we will study more accurately the mechanism of helicity generation by primordial axion oscillations before the QCD phase transition.

II. COSMIC AXION OSCILLATIONS AND MAGNETIC HELICITY

We start with the following interaction Lagrangian between the cosmic axion field ϕ and the electromagnetic field A_μ in curved space-time

$$\mathcal{L} = \sqrt{-g}\,\big(\tfrac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - \tfrac{1}{2}m_a^2\phi^2 - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{4}g_{a\gamma}\phi F_{\mu\nu}\widetilde{F}^{\mu\nu} + j^\mu A_\mu\,\big)\,, \tag{2}$$

where the external current j^μ takes into account the interaction between the cosmic plasma and the primordial magnetic field (see below).

Physics becomes more clear if we introduce the electric and magnetic fields $\mathbf{E}$ and $\mathbf{B}$. In a flat universe described by a Robertson-Walker metric, $ds^2 = dt^2 - a^2 d\mathbf{x}^2$, where $a(t)$ is the expansion parameter normalized so that at the present time t_0, $a(t_0) = 1$, this operation can be performed in the usual way as

$$F_{0i} = -aE_i, \quad F_{ij} = \epsilon_{ijk}a^2B_k, \tag{3}$$

where Latin indices range from 1 to 3. We shall work in the Coulomb gauge $A^0 = \partial_i A^i = 0$, in which Eq. (3) become $a\mathbf{E} = -\dot{\mathbf{A}}$ and $a^2\mathbf{B} = \nabla\times\mathbf{A}$, where a dot indicates the derivative with respect to the cosmic time t, and the spatial derivatives are taken with respect to co-moving coordinates.

In terms of the electric field, the external current has the form $j^\mu = (0, \sigma\mathbf{E})$, where the conductivity of primordial plasma is a temperature-dependent function that can be expressed as $\sigma(T) = \kappa T$. For $\Lambda_{\rm QCD} \lesssim T \lesssim m_W$ (where $\Lambda_{\rm QCD} \simeq 200\,$MeV and $m_W \simeq 80\,$GeV), κ is a slowly increasing function of temperature of order unity [23] ($\kappa \simeq 0.76$ for $T \simeq \Lambda_{\rm QCD}$, $\kappa \simeq 6.7$ for $T \simeq m_W$). For very high intensities of the magnetic field, that is above the critical value $B_c = m_e^2/e \simeq 4.4\times10^{13}$G ($m_e$ and e are the mass and absolute value of the electric charge of the electron), the expression for the conductivity needs to be multiplied by B/B_c [24].

Introducing the "angle" $\Theta = \phi/f_a$, the Hubble parameter $H = \dot{a}/a$, and neglecting any spatial variation of ϕ, from Lagrangian (2) we get the equations of motion

$$\ddot{\Theta} + 3H\dot{\Theta} + m_a^2\Theta = \frac{\alpha_{\rm em}}{2\pi f_a^2}\,\mathbf{E}\cdot\mathbf{B}, \tag{4}$$

$$\frac{\nabla\times\mathbf{B}}{a} = \mathbf{j} + \mathbf{j}_D + \mathbf{j}_\Theta, \tag{5}$$

where $\mathbf{j} = \sigma \mathbf{E}$ is the Ohmic current, $\mathbf{j}_D = \dot{\mathbf{E}} + 2H\mathbf{E}$ is the displacement current in the expanding universe, and we have introduced the current $\mathbf{j}_\Theta = (\alpha_{\rm em}/2\pi)\dot{\Theta}\mathbf{B}$.

It is useful to observe that in the early universe $\sigma \gg H$. Taking into account the expression for the Hubble parameter $H \simeq 1.66 g_*^{1/2} T^2/m_{Pl}$, where g_* is the total number of effectively massless degrees of freedom and m_{Pl} is the Planck mass, we get $H/\sigma \sim 10^{-18}{\rm GeV}/T$ (here we have assumed $B < B_c$, and we have taken $\kappa \sim 1$ and $g_* \sim 10^2$). Note that H/σ is even smaller for $B > B_c$.

Observing that $|\mathbf{j}_D|/|\mathbf{j}\,| \sim H/\sigma$, in Eq. (5) we can neglect the displacement current with respect to the Ohmic current. Taking the curl of Eq. (5) we then get

$$\dot{\mathbf{B}} = -2H\mathbf{B} + \frac{\nabla^2 \mathbf{B}}{\sigma a^2} + \frac{\alpha_{\rm em}}{2\pi} \frac{\dot{\Theta} \nabla \times \mathbf{B}}{\sigma a} \, . \tag{6}$$

The first term in the right hand side of the above equation describes the adiabatic dilution of the magnetic field due to the expansion of the universe, the second one takes into account the Ohmic dissipations, while the third one violates P and $CP-$symmetries, and then is responsible for the generation of a helical component of the magnetic field. Equation (6) describes the well known α^2-dynamo effect [2] (in the expanding universe) where the dynamo coefficient is $\alpha_{\rm dyn} = (\alpha_{\rm em}/2\pi)(\dot{\Theta}/\sigma)$. If the dynamo is "efficient" (i.e. if the the dynamo term in the right hand side of Eq. (5) dominates the other two) then the magnetic field is exponentially amplified, while if the dynamo is not efficient (and neglecting any dissipative effect) the magnetic field is frozen into the plasma. We will see that this is indeed our case.

It is useful to work in Fourier space and define the Fourier transform $\mathbf{B}(\mathbf{k},t)$ of the magnetic field $\mathbf{B}(\mathbf{x},t)$ according to $\mathbf{B}(\mathbf{k},t) = \int d^3x \, e^{-i\mathbf{k}\cdot\mathbf{x}} \, \mathbf{B}(\mathbf{x},t)$, where $\mathbf{x}$ and $\mathbf{k}$ are co-moving coordinates and wavenumbers, respectively. Introducing the orthonormal helicity basis $\{\mathbf{e}_+, \mathbf{e}_-, \mathbf{e}_3\}$, where $\mathbf{e}_\pm = (\mathbf{e}_1 \pm i\mathbf{e}_2)/\sqrt{2}$, $\mathbf{e}_3 = \mathbf{k}/k$ (with $k = |\mathbf{k}|$), and $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ form a right-handed orthonormal basis, the magnetic field can be decomposed as $\mathbf{B} = B_+\mathbf{e}_+ + B_-\mathbf{e}_-$, where B_+ and B_- represent the positive and negative helicity components of $\mathbf{B}$, respectively. In this basis Eq. (6) becomes

$$\dot{B}_\pm = -2HB_\pm - \frac{k^2 B_\pm}{\sigma a^2} \pm \frac{\alpha_{\rm em}}{2\pi} \frac{\dot{\Theta} k B_\pm}{\sigma a} \, . \tag{7}$$

It is clear from the above equation that the two helicity states of the magnetic field evolve differently due to the P and $CP-$violating dynamo term. The solutions of Eq. (7) are easily found

$$B_\pm(k,t) = B_\pm(k,t_i) \left(\frac{a_i}{a}\right)^2 \exp(-k^2 \ell_d^2 \mp k\ell_\Theta), \tag{8}$$

where t_i is an arbitrary time, $a_i = a(t_i)$, and we have defined the co-moving dissipation and dynamo lengths

$$\ell_d^2(t) = \int_{t_i}^t \frac{dt}{\sigma a} \, , \quad \ell_\Theta(t) = -\frac{\alpha_{\rm em}}{2\pi} \int_{t_i}^t dt \, \frac{\dot{\Theta}}{\sigma a} \, . \tag{9}$$

It is useful to introduce the spectra of the magnetic energy and the magnetic helicity

$$\mathcal{E}_B(k,t) = \left(\frac{k}{2\pi}\right)^2 \left(|B_+|^2 + |B_-|^2\right), \tag{10}$$

$$\mathcal{H}_B(k,t) = a^4 \frac{k}{2\pi^2} \left(|B_+|^2 - |B_-|^2\right), \tag{11}$$

so that the magnetic energy and helicity are given by

$$E_B(t) = \frac{1}{2}\int d^3x \, \mathbf{B}^2 = \int dk \, \mathcal{E}_B(k,t), \tag{12}$$

$$H_B(t) = a^2 \int d^3x \, \mathbf{A}\cdot\mathbf{B} = \int dk \, \mathcal{H}_B(k,t). \tag{13}$$

It is clear from Eqs. (10) and (11) that any magnetic field configuration satisfies the realizability condition $|\mathcal{H}_B(k,t)| \leq 2a^4 k^{-1} \mathcal{E}_B(k,t)$. Now, defining $\mathcal{H}_B^{\rm max}(k,t) = 2a^4 k^{-1} \mathcal{E}_B(k,t)$, and inserting Eq. (8) in Eqs. (10) and (11), we get

$$\mathcal{E}_B(k,t) = \mathcal{E}_B(k,t_i) \left(\frac{a_i}{a}\right)^4 \exp(-2k^2\ell_d^2) \cosh(2k\ell_\Theta), \tag{14}$$

$$\mathcal{H}_B(k,t) = -\mathcal{H}_B^{\rm max}(k,t_i) \exp(-2k^2\ell_d^2) \sinh(2k\ell_\Theta), \tag{15}$$

where we have supposed that the initial helicity is null. We shall assume that the initial magnetic energy spectrum can be represented by the following simple function

$$\mathcal{E}_B(k,t_i) = \lambda_B k^p \exp(-2k^2 \ell_B^2), \tag{16}$$

where λ_B and ℓ_B are constants. For $k \ll 1/\ell_B$, the magnetic energy spectrum possesses a power law behavior, [1] while for large k the spectrum is suppressed exponentially in order to have a finite energy. Also, we introduce the so-called correlation length $\xi_B(t)$ defined by

$$\xi_B(t) = a(t) \, \frac{\int dk \, k^{-1} \mathcal{E}_B(k,t)}{\int dk \, \mathcal{E}_B(k,t)} \, . \tag{17}$$

The constant ℓ_B is then related to the initial correlation length $\xi_B(t_i)$ by

$$\ell_B = \frac{\Gamma((1+p)/2)}{\sqrt{2}\,\Gamma(p/2)} \, \frac{\xi_B(t_i)}{a_i} \, , \tag{18}$$

where $\Gamma(x)$ is the Euler gamma function. Then, ℓ_B is proportional to the co-moving initial correlation length.

We now return to the evolution equation for the Θ-angle. Assuming that the back-reaction [that is the right-hand side of Eq. (4)] of the electromagnetic field

[1] Analyticity of the magnetic field correlator $\langle B_i(\mathbf{r}_1) B_j(\mathbf{r}_2) \rangle$ defined on a compact support forces the spectral index p to be even and equal or larger than 4 [25]).

is negligible (this will be justified *a posteriori*), the evolution of the cosmic axion field follows the standard description [19]. Here, we just remember the expression for the temperature-dependent axion mass, $m_a(T) \simeq 0.1\, m_a(0)(T/\Lambda_{\rm QCD})^{-3.7}$, valid for $\pi T/\Lambda_{\rm QCD} \gg 1$, where the axion mass at zero temperature is $m_a(0) \simeq 0.6\,(10^7{\rm GeV}/f_a)\,{\rm eV}$. The temperature at which the axion field starts to oscillate is usually indicated by T_1, and it is found by solving $m_a(T_1) = 3H(T_1)$. Its value is given by $T_1/{\rm GeV} \simeq 0.9 f_{12}^{-0.18}\Lambda_{200}^{0.65}$, where $f_{12} = f_a/(10^{12}{\rm GeV})$ and $\Lambda_{200} = \Lambda_{\rm QCD}/(200{\rm MeV})$.
Introducing the normalized time $\tau = t/t_1$, the equation of motion for the Θ-angle becomes

$$\Theta'' + \frac{3}{2\tau}\,\Theta' + \frac{9}{4}\,\tau^{3.7}\Theta = 0, \qquad (19)$$

where a prime indicates a derivative with respect to τ. In Fig. 1, we plot the solution of the above equation, where $\Theta_i = \Theta(t_i)$ and $\dot\Theta(t_i) = 0$.

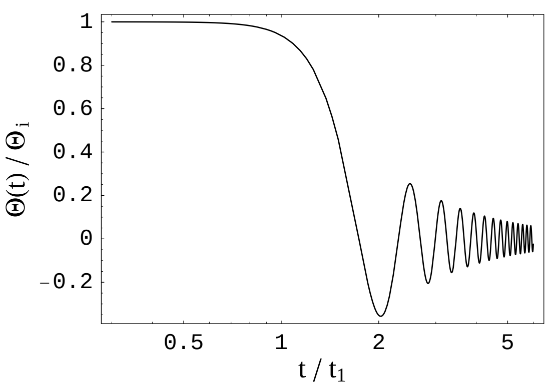

FIG. 1: $\Theta(t)$ as a function of t/t_1.

Inserting Eqs. (14) and (15) in Eqs. (12) and (13) we find, respectively

$$\bar B^2(t) = \bar B_i^2 \left(\frac{a_i}{a}\right)^4 \left(\frac{\ell_B^2}{\ell_B^2+\ell_d^2}\right)^{(1+p)/2} \times {}_1F_1\!\left(\frac{1+p}{2}, \frac{1}{2}; \frac{\ell_\Theta^2/2}{\ell_B^2+\ell_d^2}\right), \qquad (20)$$

$$H_B(t) = -2(2\pi)^3 a_i^4 \bar B_i^2 \left(\frac{\ell_B^2}{\ell_B^2+\ell_d^2}\right)^{(1+p)/2} \times {}_1F_1\!\left(\frac{1+p}{2}, \frac{3}{2}; \frac{\ell_\Theta^2/2}{\ell_B^2+\ell_d^2}\right)\ell_\Theta, \qquad (21)$$

where $\bar B_i = \bar B(t_i)$, ${}_1F_1(a,b;x)$ is the Kummer confluent hypergeometric function [26], and we have introduced the root-mean-square value of the magnetic field

$$\bar B^2 = \int \frac{d^3k}{(2\pi)^3}\frac{d^3k'}{(2\pi)^3}\,\mathbf{B}(\mathbf{k})\cdot\mathbf{B}(\mathbf{k}') = \frac{2E_B}{(2\pi)^3}\,. \qquad (22)$$

From Eq. (20) we see that the dynamo is efficient (and then there is exponential growth of the magnetic field) if the dynamo length ℓ_Θ is greater than both the dissipation length ℓ_d and the initial correlation length ℓ_B.
Magnetic fields whose correlation lengths are smaller than the dissipation length and for which the dynamo is not efficient are quickly dissipated. This is easily seen from the evolution equation for the magnetic field, where the suppression factor due to dissipation is equal to the third term on the right-hand side of Eq. (20). Because we are supposing that the magnetic field interacting with the axion is ultimately the field that we observe today in the galaxies and galaxy clusters, we assume that the magnetic field is in the inertial regime, $\ell_d \ll \ell_B$, during all its evolution.

Since, in general, the conductivity depends on the value of $\bar B$, to know the behavior of the dynamo length ℓ_Θ we have to solve Eq. (20) which is a integral equation for $\bar B$. To this end, we re-write ℓ_Θ as

$$\ell_\Theta(\tau) = \frac{\alpha_{\rm em}}{2\pi}\frac{\Theta_i}{\sigma_1 a_1}\,f(\tau), \qquad (23)$$

where the subscript "1" indicates that the quantity is calculated at $t = t_1$, and we have introduced the function $f(\tau) = -\int_{\tau_i}^{\tau} d\tau \tau^{-1/2}(\Theta'/\Theta_i)(\sigma_1/\sigma)$, with $\tau_i = t_i/t_1$. We have solved numerically Eq. (20) for $f(t)$ taking, for purpose of simplicity, $\sigma = T(1+\bar B/B_c)$, which smoothly interpolates between the two limiting cases $\bar B \ll B_c$ and $\bar B \gg B_c$. In Fig. 2, we plot $f(t)$ as a function of t/t_1, where Θ is the solution of Eq. (19), for three different cases of initial strength of the magnetic field. Here, we have parameterized the initial magnetic field as $\bar B_i = bT_i^2$, so that we have $\bar B_i \simeq 1.5\times10^{19} b\,(T_i/{\rm GeV})^2{\rm G}$. [If we force the magnetic energy density to be less than the energy density of the universe, given in the radiation era by $\rho = (\pi^2/30)g_* T^4$, we get $b \lesssim 0.1 g_*(T_i)^{1/2}$.]

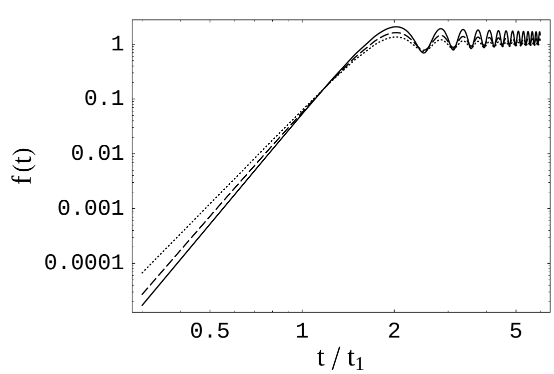

FIG. 2: $f(t)$ as a function of t/t_1 for $T_i = 10\,{\rm GeV}$, $\Theta_i = 1$, $\bar B_i = bT_i^2$, $\xi_B(T_i) = 10^{-5} d_H(T_i)$, and $p = 4$. Solid line: $b = 10^{-1}$, corresponding to $\bar B_i \simeq 10^{20}{\rm G}$; Dashed line: $b = 10^{-6}$, corresponding to $\bar B_i \simeq 10^{15}{\rm G}$; Dotted line: $b = 10^{-10}$, corresponding to $\bar B_i \simeq 10^{11}{\rm G}$.

We checked that, varying the initial conditions for the magnetic field [i.e. T_i, b, $\xi_B(T_i)$, and p], the characteristic behavior of $f(t)$ in Fig. 2, that is $f(t) \simeq 0$ for $t < t_1$, and $f(t) \simeq 1$ for $t > t_1$, does not change. This is easily understood if we approximate (see Fig. 1) the Θ-angle as $\Theta(t)/\Theta_i \simeq 1-\theta(t-t_1)$, where $\theta(t)$ is the step-function. In this case, it turns out that f is a step-function centered at $t = t_1$, $f(t) \simeq \theta(t - t_1)$.

The dynamo length ℓ_Θ is small compared to the dissipative length (and then is smaller than ℓ_B which we assumed much greater than ℓ_d). In fact, for $t > t_1$ and taking $f \sim 1$, we get $\ell_\Theta/\ell_d \sim 10^{-5}\Theta_i/\tau^{1/2}$ (where we have taken $\bar{B} < B_c$, and then the value of ℓ_Θ/ℓ_d is even smaller for magnetic fields for which $\bar{B} > B_c$). Then, from Eqs. (20) and (21), we get that in the inertial regime the magnetic field evolves as $\bar{B} \propto a^{-2}$ (i.e. the field is frozen into the plasma) while the magnetic helicity is

$$H_B(t) \simeq -2(2\pi)^3 a_i^4 \bar{B}_i^2 \ell_\Theta. \tag{24}$$

Finally, taking $\ell_\Theta \simeq (\alpha_{\rm em}/2\pi)(\Theta_i/\sigma_1 a_1)$ for $t > t_1$, we get that the magnetic helicity remains constant after its generation (say at $t \sim t_1$) and is equal to

$$H_B \simeq -b^2 \Theta_i \, (10^{-17}{\rm G})^2 \, {\rm kpc} \tag{25}$$

for $\bar{B}_1 < B_c$, corresponding to $b \lesssim 10^{-6}({\rm GeV}/T_1)^2$, and

$$H_B \simeq -b \, \Theta_i \left(\frac{\rm GeV}{T_1}\right)^2 (10^{-20}{\rm G})^2 \, {\rm kpc} \tag{26}$$

in the case $\bar{B}_1 > B_c$, or $b \gtrsim 10^{-6}({\rm GeV}/T_1)^2$.
Since we expect that Θ_i is of order unity, and because $T_1 \simeq 1\,{\rm GeV}$, we conclude that for very strong magnetic fields (corresponding to $b \simeq 1$), the generated magnetic helicity at $t \sim t_1$ is about $|H_B| \simeq (10^{-20}{\rm G})^2$ kpc.

We can now justify *a posteriori* the assumption that the back-reaction of the electromagnetic field on the evolution of Θ is negligible. [2] Integrating Eq. (4) with respect to d^3x, observing that $\dot{H}_B = -2a^3 \int d^3x \, \mathbf{E} \cdot \mathbf{B}$, and taking into account Eqs. (9) and (24), we obtain

$$\ddot{\Theta} + (3H + \varepsilon)\dot{\Theta} + m_a^2 \Theta = 0, \tag{27}$$

where we have defined $\varepsilon = 2\pi\alpha_{\rm em}^2 (a_1/a)^4 (\bar{B}_1^2/f_a^2 \sigma)$. Writing $\bar{B}_1 = bT_1^2$, and assuming $f_a \sim 10^{12}\,{\rm GeV}$ and $\bar{B} < B_c$, we get $\varepsilon/H \sim 10^{-10} b^2 T/{\rm GeV}$ (ε is even smaller for $\bar{B} > B_c$). The smallness of ε justifies our previous assumption of neglecting the back-reaction of the electromagnetic field on the cosmic evolution of the axion.
We finally note that the parameter ε measures the rate of the axion condensate evaporation. Indeed, Eq. (27) can be written as

$$\frac{d(\ln n_a)}{dt} = -\varepsilon, \tag{28}$$

where n_a is the number of axions per co-moving volume. Thus, we see that the presence of an external magnetic field opens a new channel for the dissipation of the axion condensate, though this dissipation is very small (for other dissipation mechanisms of the axion condensate see [27] and references therein).

[2] Below the QCD phase transition, when the axion mass is constant and equal to its zero temperature value, the fact that a primordial magnetic field does not interfere with the cosmic evolution of the axion has been shown in Ref. [21].

III. CONCLUSIONS

In this paper we have studied in detail the axion-magnetic field system during the period of primordial axion oscillations ($T \sim 1$ GeV). Here we summarize our main results and give some perspectives.

The presence of a primordial magnetic field at the time when the cosmological axion oscillations begin amplifies the axion decay probability and, contemporaneously, catalyzes the photon production (enlarging the magnetic energy itself). The effect of this process on the axion field is minimal, since the number of decaying axions is rather small $\Delta n_a/n_a \sim 10^{-10}$. A simple consequence is that the cosmological bound on the axion mass is preserved, even in the presence of a very strong external magnetic field.

However, the axion oscillations leave a *signature* on the magnetic field itself, in the form of magnetic helicity. This is due to the different probability for creating photons of different helicities. Thus the axion oscillations, interacting with a magnetic field, originate a macroscopic $CP-$odd state. The amount of helicity produced depends on the initial intensity of the magnetic field. At late times, the turbulence of the primordial plasma should be taken into account in considering the evolution of both magnetic energy and helicity densities. However, it is now believed that helicity is a quasi-conserved quantity in magnetohydrodynamic turbulence [5, 28]. Hence, the helicity generated in the primordial universe should survive until today without changing its major properties, and could be important for the dynamo mechanism of magnetic fields operating in galaxies and clusters of galaxies (see, e.g., Ref. [1]). Moreover, the presence of a helical component may speed up the growth of the magnetic field correlation length during its evolution in the primordial universe through the so-called inverse cascade of magnetic energy [29], leading today to magnetic fields on larger scales [8, 30].

In principle, a helical magnetic field could leave peculiar imprints on the Cosmic Microwave Background Radiation (CMBR). Unfortunately, the maximal helicity producible in our mechanism, $|H_B| \simeq (10^{-20}{\rm G})^2$ kpc, is much smaller than that detectible in near future CMBR experiments, which is of order of $(10^{-9}{\rm G})^2$ Mpc [31].

However, from the above discussion it emerges a strong relation between axion and magnetic helicity. We believe that this subject deserves further investigations.

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